Received: *July 20, 1998,* Accepted: *July 22, 1998*

The large N limit of $\mathcal{N} = 2, 1$ field theories from threebranes in F-theory

Ofer Aharony

Department of Physics, Rutgers University, Piscataway, NJ 08855, USA
oferah@physics.rutgers.edu

Ansar Fayyazuddin and Juan Maldacena

Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA
ansar@schwinger.harvard.edu, malda@bose.harvard.edu

Abstract: We consider field theories arising from a large number of D3-branes near singularities in F-theory. We study the theories at various conformal points, and compute, using their conjectured string theory duals, their large N spectrum of chiral primary operators. This includes, as expected, operators of fractional conformal dimensions for the theory at Argyres-Douglas points. Additional operators, which are charged under the (sometimes exceptional) global symmetries of these theories, come from the 7-branes. In the case of a D_4 singularity we compare our results with field theory and find agreement for large N. Finally, we consider deformations away from the conformal points, which involve finding new supergravity solutions for the geometry produced by the 3-branes in the 7-brane background. We also discuss 3-branes in a general background.

Keywords: Duality in Gauge Field Theories, 1/N Expansion, D-branes, Brane Dynamics in Gauge Theories.

JHEP07(1998)013

Contents

1. Introduction

Through the conjecture [1, 2, 3] relating field theories with string theory / M theory solutions, it has become possible to explore the large N limit of field theories by using supergravity solutions. One can motivate this relationship by starting with a brane configuration in the full string theory and then taking a low energy limit which decouples the field theory from gravity, and at the same time considering the near horizon geometry of the corresponding black brane supergravity solution.

In this paper we study $\mathcal{N} = 2$ and $\mathcal{N} = 1$ field theories which arise as worldvolume theories of D3-branes moving near 7-branes, i.e. 3-branes in F-theory [4, 5]. In the simplest case F-theory reduces to a perturbative $\mathbb{Z}_2$ orientifold [28]. The low-energy theory on the worldvolume of N D3-branes is then [6, 8, 9] a $USp(2N)$ $\mathcal{N} = 2$ gauge theory with an antisymmetric and four fundamental hypermultiplets. Deformations away from the orientifold correspond to giving some masses to the fundamentals. When the masses are zero and the 3-branes are at the origin (the orientifold point) we have a non-trivial conformal field theory, whose large N limit can be calculated by making an orientifold of the $AdS_5 \times S^5$ gravity background. Note that this theory has dynamical quarks in the fundamental representation, so it can be used to study their behavior in the large N limit. This system was studied in [10], together with other $\mathbb{Z}_n$ orbifolds which do not have a perturbative limit (E_6, E_7, E_8) that arise as we bring together many nonlocal 7-branes.

In this paper we study these configurations directly, without using their orientifold description. In addition we can study conformal field theories that occur at Argyres-Douglas points [11, 12]. These cannot be viewed as orbifolds but nevertheless we are able to find the large N gravity description for them. One of the novel features is that these theories have operators with fractional conformal dimensions, which arise via some simple generalization of spherical harmonics. One interesting implication of these states is that we are forced to consider the multiple cover of AdS_5. Though this is necessary in general to avoid closed timelike loops, the energies of states found in previous computations, corresponding to operators with integer (or 1/2 integer) dimensions, were consistent with periodic time.

In general, when the supergravity solution is singular, there will be states coming both from the bulk fields and from the fields living on the singularities. When the low-energy theory on the singularities is weakly coupled, these states may be analyzed as easily as the bulk modes, but this has not been done until now (except for a discussion of tensor multiplet fields in [32]). In the theories we discuss there will always be operators coming from states localized on the singular surfaces, which in the orientifold case correspond to the fixed surfaces. Topologically these singular surfaces look like an S^3 inside the S^5. We analyze the contributions to the spectrum from the singularities as well as those of the bulk modes, checking them against the expectations from the weakly coupled limit when it exists, and finding agreement for large N. Similarly, one can construct and analyze configurations with two (or three) sets of intersecting 7-branes, which give rise to $\mathcal{N} = 1$ superconformal field theories [13, 8] about which not much is known. Related configurations, that also describe conformal field theories, were studied in [14].

Finally, we consider an arbitrary configuration of 7-branes. It has been shown in [6, 8, 9] that (for theories with $\mathcal{N} = 2$ SUSY) the metric seen by 3-brane probes moving in the 7-brane background is exactly the low-energy effective action for the corresponding field theory. We now go beyond the probe approximation and consider the metric produced by the 3-branes. We find that we can reduce the full non-linear gravity equations to a single Laplace equation on the metric produced by the 7-branes. We discuss some qualitative features of the solution which do not require solving the equation (a problem left for the future). These solutions provide examples of four dimensional non-conformal field theories which can be solved in the large N limit via supergravity. In particular one can have an asymptotically free theory with a logarithmic running of the coupling.

In section 2 we review some aspects of the 7-brane geometry. We begin by analyzing $\mathcal{N} = 2$ SCFTs which arise from D3-branes at 7-brane singularities. In section 3 we calculate the operators coming from the fields that live in the bulk of spacetime, while in section 4 we calculate the contribution of the singular surfaces (the 7-branes). In

JHEP07(1998)013

section 5 we sketch the analysis for $\mathcal{N}=1$ theories, and in section 6 we consider the supergravity solution in the general non-conformal case.

2. D3-branes and sevenbranes

We consider the field theory arising when D3-branes move close to 7-branes. In the field theory limit the 3-branes and 7-branes are at substringy distances, so only the behavior near the singularities, corresponding to the local F-theory geometry, is important. The simplest non-trivial case is when the 7-branes correspond to a $\mathbb{Z}_2$ orientifold point [28]. A single 3-brane moving near a $\mathbb{Z}_2$ orientifold point corresponds to a $USp(2) \equiv SU(2)$ theory with four flavors [6]. When we have N branes we have [8, 9] a $USp(2N)$ theory with one antisymmetric and four fundamental hypermultiplets. If all the 7-branes and the 3-branes are sitting together we have (at low energies) a superconformal field theory. Since there is an $SO(8)$ gauge field living on the 7-branes we have an $SO(8)$ global symmetry on the D3-branes. This $\mathbb{Z}_2$ singularity is called D_4.

We can get other theories by looking at different configurations of the 7-branes. At generic points in 7-brane moduli space we will get non-conformal field theories on the D3-branes. These theories will be the subject of the last section. There are 7 types of singularities which have a constant value of the dilaton and give rise to conformal field theories on the D3-branes. They are the Argyres-Douglas points H_0, H_1 and H_2 [11, 12] which have A_0, A_1 and A_2 gauge theories on the 7-branes (and corresponding global symmetries on the D3-branes), and the D_4, E_6, E_7 and E_8 singularities, which give rise to corresponding gauge and global symmetries [15]. Only the D_4 singularity can occur for any value of the dilaton; the other singularities occur at a fixed value of the string coupling, of order one.

The gravity description D3-branes near 7-brane singularities was derived in [10] for orbifold cases (D_4, E_6, E_7, E_8), and a similar description applies to all types of singular 7-branes. For a D3-brane near a single 7-brane singularity, the resulting metric is similar to the $AdS_5 \times S^5$ metric, but with the metric of the compact space, $d\tilde{\Omega}_5^2$, given by the angular variables of

$$ds^2 \equiv dr^2 + r^2 d\tilde{\Omega}_5^2 = |dz|^2/|z|^\alpha + dx_3^2 + dx_4^2 + dx_5^2 + dx_6^2 \tag{2.1}$$

instead of the S^5 metric, and with an appropriate monodromy for the B-fields around the point $|z|=0$. The value of α depends on the singularity type; it is $\alpha = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}$ for $G = H_0, H_1, H_2, D_4, E_6, E_7$ and E_8, respectively. Defining new variables $w = z^{1-\alpha/2}$ and $\tan^2(\theta) = |w|^2/(x_3^2+x_4^2+x_5^2+x_6^2)$, we can write the angular part of (2.1) as

$$ds^2 = d\theta^2 + \sin^2(\theta)d\phi^2 + \cos^2(\theta)d\Omega_3^2\,, \tag{2.2}$$

where $d\Omega_3^2$ is the metric on S^3, $0 \leq \theta \leq \pi/2$ and $\phi = \arg(w)$ is periodic with period $2\pi(1-\alpha/2)$, which corresponds to a full rotation in the z plane. The B-fields generally have $SL(2,\mathbb{Z})$ monodromies around this circle [28]; for D_4, E_6, E_7 and E_8 these are described in [15], and the H_0, H_1 and H_2 cases are similar in this respect to E_8, E_7 and E_6, respectively. Except in the D_4 case, the string coupling τ also has monodromies around this circle, which can be treated in a similar fashion. The changed periodicity and monodromies are the only difference between this solution and the S^5 solution, and it means that we have a G-type singularity at $\sin(\theta) = 0$, which is an S^3 in the compact space. This can be interpreted as a G-type 7-brane sitting at $\sin(\theta) = 0$ (and filling AdS_5). The presence of the 7-branes breaks the $SO(6)$ isometry symmetry (which is identified with the global symmetry in the four dimensional field theory) to $SO(4) \times SO(2) \simeq SU(2)_R \times SU(2)_L \times U(1)_R$.

The low-energy spectrum of the type IIB string theory on this space will have a contribution from the bulk SUGRA modes on this space, and a contribution from the singularities. We will calculate both of these contributions below.

Similarly, for the (orthogonal, SUSY-preserving) intersection of two 7-brane singularities [16], we will find the metric corresponding to the angular variables of

$$ds^2 = |dz_1|^2/|z_1|^{\alpha_1} + |dz_2|^2/|z_2|^{\alpha_2} + dx_5^2 + dx_6^2 \,, \tag{2.3}$$

where α_1, α_2 are the values of α corresponding to each of the singularities. We will describe this in more detail below. Notice that the values of α_i have to be compatible with one another if both singularities exist only at fixed values of the dilaton. In this case the isometry symmetry is broken to $U(1)^3$.

3. Chiral primaries of $\mathcal{N} = 2$ SCFTs from bulk modes

In this section we will compute the contribution to the low-energy spectrum on AdS_5 from the supergravity modes, generalizing the analysis of [10]. As discussed above, these modes propagate on a space which has the metric of an S^5 but with unusual boundary conditions on one of the angular variables (denoted by ϕ in (2.2)). Instead of being periodic with period 2π, this variable is periodic with period $2\pi(1-\alpha/2)$, with a monodromy relating the fields at $\phi = 0$ and at $\phi = 2\pi(1-\alpha/2)$. As discussed in [10], the action of this monodromy on the supergravity fields can be diagonalized, in which case it corresponds to a phase acting on some of the fields. We will start by discussing scalar fields on the compact space with no non-trivial monodromy (for instance, the graviton fields $h_{\mu\nu}$ and h_a^a), and discuss the general case later.

The spectrum of the supergravity fields is determined by expanding the supergravity equations (coupled to the 7-brane fields) to linearized order in the fluctuations around the configuration described above. In the large g^2N limit, the sphere is large, and we can ignore the interactions of these fields with the 7-brane fields which live on a codimension two subspace of the compact space. Since the metric on the compact space is the same as the S^5 metric, we find the same equations as in the S^5 reduction of the supergravity fields [17]. The only difference is the different periodicity of the fields in the ϕ direction.

For massless scalar fields, the linearized equation of motion is simply the Laplace equation on the compact space, whose solutions (for S^5) are the scalar spherical harmonics, in k'th symmetric traceless representations of $SO(6)$ with eigenvalue $k(k+4)$. It is not hard to derive this result also using the metric (2.2) (with standard boundary conditions). The Laplacian in this metric is

$$\nabla_5^2 = \frac{1}{\sin\theta\cos^3\theta}\frac{d}{d\theta}\sin\theta\cos^3\theta\frac{d}{d\theta} - \frac{m^2}{\sin^2\theta} - \frac{l(l+2)}{\cos^2\theta}\,, \tag{3.1}$$

where m^2 and $l(l+2)$ are the eigenvalues of the Laplacian on S^1 and S^3 respectively. The latter has eigenfunctions in the $(\mathbf{l+1},\mathbf{l+1})$ representations of $SU(2)_L\times SU(2)_R$, with $l=0,1,2,\cdots$. The $U(1)_R$ charge of the field is $2m$, and m is an integer for standard boundary conditions. Plugging in these eigenvalues we find a hypergeometric equation in terms of $\cos^2(\theta)$, which has a discrete series of solutions that are regular at $\theta=0$ and at $\theta=\pi/2$, labeled by $n=0,1,2,\cdots$. The total Laplacian eigenvalue corresponding to the n'th solution is $k(k+4)$ where $k=|m|+l+2n$. For a particular value of k the states we find fill in the k'th symmetric traceless multiplet of $SO(6)$. For instance, for $k=2$ we find a $(\mathbf{1},\mathbf{1})_{\pm\mathbf{4}}$ representation from $m=\pm2, l=0, n=0$, a $(\mathbf{1},\mathbf{1})_{\mathbf{0}}$ representation from $m=0, l=0, n=1$, a $(\mathbf{2},\mathbf{2})_{\pm\mathbf{2}}$ representation from $m=\pm1, l=1, n=0$ and a $(\mathbf{3},\mathbf{3})_{\mathbf{0}}$ representation from $m=0, l=2, n=0$. Together these form the $\mathbf{20}'$ of $SO(6)$.

How does this change when we implement the different periodicity conditions described above ? The only thing that changes is that we now have different possible values of m from the S^1 Laplacian. Instead of having m be an integer from the eigenfunctions $e^{im\phi}$, we now have $m=\tilde{m}/(1-\alpha/2)$ for integer $\tilde{m}$, corresponding to the eigenfunctions $e^{i\tilde{m}\phi/(1-\alpha/2)}$. Otherwise, the analysis is the same as above. For the analysis of the hypergeometric equation it does not matter if m is an integer or not. Thus, for any integer $\tilde{m}$ and non-negative integers l and n we will find an eigenfunction of the Laplacian in the $(\mathbf{l+1},\mathbf{l+1})_{2\tilde{m}/(1-\alpha/2)}$ representation, with an eigenvalue $k(k+4)$, where

$$k = \frac{|\tilde{m}|}{(1-\alpha/2)} + l + 2n\,. \tag{3.2}$$

The AdS/CFT correspondence relates a massless field to an operator of scaling dimension $\Delta=k+4$. Some of the operators that were originally (in the $\mathcal{N}=4$ theory) in chiral primary multiplets will remain in chiral primary multiplets also of the $\mathcal{N}=2$ superconformal algebra, while others can be in large (non-chiral) multiplets of the $\mathcal{N}=2$ algebra. It is easy to identify which scalar fields are lowest components of chiral primary multiplets of the $\mathcal{N}=2$ algebra, since these fields obey $\Delta=2j+R/2$ where j is the spin of their $SU(2)_R$ representation and R is their $U(1)_R$ charge. Fields that do not obey this relation can be either descendants of the chiral primary fields or non-chiral fields.

Similarly, for scalar fields which have some phase from the monodromy, again the only difference from the S^5 case will be the appearance of different eigenvalues m in the Laplacian, which will be shifted by a constant compared to the values discussed above. As above, this will change the $U(1)_R$-charge of the corresponding fields and their dimension, but otherwise the spectrum remains unchanged. For instance, let us look at the fields arising from a linear combination of h_a^a and D_{abcd}, which in the original S^5 case gave chiral primary fields of the $\mathcal{N}=4$ superconformal algebra which were identified with $\mathrm{tr}(X^{i_1}X^{i_2}\cdots X^{i_k})$.[1] Using the relation between the dimensions the R-symmetry representations, it is easy to see that the condition for this state to give a chiral primary field of the $\mathcal{N}=2$ superconformal algebra (namely, the field with lowest dimension in an $\mathcal{N}=2$ chiral primary multiplet) is that $n=0$. Consider the fields with $l=n=0$. In the original S^5 case there was no mode with $k=0$ and the mode with $k=1$ could be gauged away [17] so we only had $m=\pm 2,\pm 3,\cdots$. In the new metric it seems that $\tilde{m}=\pm 1$ is also allowed (since it no longer corresponds to a dimension one operator which would be a singleton field). Thus, we find a chiral primary field in the $(\mathbf{1},\mathbf{1})_{\mathbf{2\tilde{m}/(1-\alpha/2)}}$ representation, with dimension $\Delta=k=|\tilde{m}|/(1-\alpha/2)$, for any non-zero integer $\tilde{m}$. We can identify these fields with the natural coordinates of the Coulomb branch of these theories. For $N=1$, the field with $\tilde{m}=1$ can be identified (in the usual sense of the AdS/CFT correspondence) with the standard coordinate u on the Coulomb branch, which was shown in [12] to indeed have dimension $\Delta=1/(1-\alpha/2)$ (namely, $\Delta=6/5$ for the H_0 case, $\Delta=4/3$ for H_1, $\Delta=3/2$ for H_2, $\Delta=2$ for D_4, $\Delta=3$ for E_6, $\Delta=4$ for E_7 and $\Delta=6$ for E_8). The fields with higher values of $\tilde{m}$ may be identified with the natural coordinates on the N'th symmetric product of the u plane (for large N), which is the Coulomb branch for the theory arising from N D3-branes. Thus, our results agree with the field theory expectations in this case. Similarly, we can analyze all the other supergravity fields, and obtain predictions for the full large N spectrum of chiral primaries in all of these theories.

Note that our method of analyzing the spectrum is different from the method of [10], and apriori the results seem to be different. In particular, in [10] the spectrum is described as a projection of the original S^5 spectrum, while here we describe it as a shifted version of the same spectrum. However, it is easy to check that in the cases that correspond to orientifolds (namely, $\alpha=1,\frac{4}{3},\frac{3}{2}$ and $\frac{5}{3}$) the results actually agree. For instance, let us analyze the modes of a periodic massless scalar field which correspond to operators of dimension $\Delta=k+4$, for the case $\alpha=1$ which is a $\mathbb{Z}_2$ projection. In the analysis of [10], we start with the original supergravity spectrum [17] in which we have such a mode in the $(\mathbf{l+1},\mathbf{l+1})_{\mathbf{2m}}$ representation for $k=|m|+l+2n$, and then we project out the modes which have odd values of m, so we find a field in the $(\mathbf{l+1},\mathbf{l+1})_{\mathbf{4m}}$ representation for $k=2|m|+l+2n$ for any integer m and non-negative integers l and n. In our analysis, since in this case $m=2\tilde{m}$ for integer $\tilde{m}$, we find exactly the same result by the method described above. The same comparison works for all the other fields as well. However, the analysis above applies also to cases which do not have an orientifold description, such as the Argyres-Douglas points.

[1] Note that the equation of motion for this field includes also a linear (mass) term, causing the corresponding dimensions to be $\Delta=k$ instead of $\Delta=k+4$ [17, 3].

4. Chiral primaries of $\mathcal{N} = 2$ SCFTs from 7-brane fields

The compactifications of type IIB string theory which correspond to $\mathcal{N} = 2$ SCFTs include, as described in §2, 7-branes wrapped around an S^3 inside the compact space (and filling the AdS_5 space). For the theory corresponding to N D3-branes at a G-type singularity, the low-energy theory on these 7-branes is a 7+1 dimensional $\mathcal{N} = 1$ SYM theory of gauge group G (in order to get a conformal theory on the 3-brane the singularity can be of type $G = H_0, H_1, H_2, D_4, E_6, E_7$ or E_8). The low-energy spectrum of this compactification will thus include the Kaluza-Klein modes of this vector multiplet on $S^3 \times AdS_5$, and by the usual AdS/CFT correspondence [1, 2, 3] these will correspond to primary operators in the SCFT. Since all these operators are in small multiplets (with spins up to one), the corresponding operators will necessarily be chiral primaries, and the AdS/CFT correspondence implies that these will be all the chiral primaries charged under the global symmetry group G which remain at finite dimension in the large N limit (for the D_4 theory we need to take also large g^2N; for the other theories g_s is of order one so this is guaranteed).

In principle, we should compute the spectrum of masses of these states by linearizing the equations of motion of the 7-brane fields in the appropriate background, as was done for the supergravity fields in [17]. However, since these states are all in small multiplets, their masses (related to the dimensions of the corresponding chiral primary operators in the SCFT) are completely determined in terms of their R-symmetry representation. Thus, all we really need to do is compute which representations arise. The R-symmetry in the $\mathcal{N} = 2$ superconformal algebra is $SU(2)_R \times U(1)_R$, and a (scalar) chiral primary field in an $SU(2)_R$ representation of spin j and with $U(1)_R$ charge R has dimension $\Delta = 2j + R/2$ (where R is normalized so that the SUSY generators have $R = 1$). In the supergravity solution, this R-symmetry is part of the isometry group of the compact space, which is $SO(4) \times SO(2) \simeq SU(2)_R \times SU(2)_L \times U(1)_R$.

The 7+1 dimensional $\mathcal{N} = 1$ vector multiplet contains a vector field A_μ, a complex scalar field z and fermions. We will concentrate here only on the bosonic fields (obviously the fermionic spectrum is related to this by the supersymmetry). The field z is in the $(\mathbf{1}, \mathbf{1})_{\mathbf{2}}$ representation of the global symmetry group $SU(2)_R \times SU(2)_L \times U(1)_R$. When we put this theory on $S^3 \times AdS_5$, the vector field will decompose into an AdS_5-vector field A_μ and an S^3 vector A_a. The spectrum of fields on AdS_5 will include the KK modes of all these fields.

For the scalar field z the expansion is $z(x, y) = \sum_k z_k(x) Y^k(y)$ where x denotes the AdS_5 coordinates, y denotes the S^3 coordinates, and $Y^k(y)$ are the scalar spherical harmonics on S^3. On S^d, these are in $SO(d + 1)$ representations corresponding to symmetric traceless products of $(\mathbf{d} + \mathbf{1})$'s. For S^3 we thus find that the fields $z_k(x)$ are in the $(\mathbf{k}, \mathbf{k})_{\mathbf{2}}$ representations of the global symmetry group, for $k = 1, 2, 3, \cdots$.

For the vector field we have a similar expansion $A_\mu(x, y) = \sum_k A^k_\mu(x) Y^k(y)$, leading to vector fields on the AdS_5 space in the $(\mathbf{k}, \mathbf{k})_{\mathbf{0}}$ representation for $k = 1, 2, 3, \cdots$. For the internal components of the gauge field we have $A_a(x, y) = \sum_k A_k(x) Y^k_a(y)$ where

Y_a^k are the vector spherical harmonics on S^3. The fields $A_k(x)$ will thus be real scalar fields in the $(\mathbf{k}, \mathbf{k+2})_\mathbf{0} + (\mathbf{k+2}, \mathbf{k})_\mathbf{0}$ representation of $SU(2)_R \times SU(2)_L \times U(1)_R$ for $k = 1, 2, 3, \cdots$. All these states are in the adjoint representation of the gauge group G, and according to the AdS/CFT correspondence they correspond to the only G-charged operators that remain at finite dimension in the large N limit.

Next, let us describe the supermultiplet structure that these fields fall into. The supercharges of the $\mathcal{N} = 2$ theory are in the $(\mathbf{2}, \mathbf{1})_\mathbf{1}$ representation of $SU(2)_R \times SU(2)_L \times U(1)_R$. There exist small representations of the superconformal group that start with a lowest component that is a real scalar in the $\mathbf{k_0}$ representation of $SU(2)_R \times U(1)_R$. The other components of the multiplet arise by acting with supercharges Q on this lowest component. We will describe here only the bosonic components. Acting with two Q's gives rise again to a scalar field, in the $(\mathbf{k-2})_\mathbf{2}$ representation. The fact that we find a smaller R-symmetry representation is a manifestation of the fact that this is a chiral primary field in a small representation. Acting with two $\bar{Q}$'s gives the complex conjugate field. Acting with one Q and one $\bar{Q}$ gives a vector field in the $(\mathbf{k-2})_\mathbf{0}$ representation. Finally, acting with two Q's and two $\bar{Q}$'s gives a scalar field in the $(\mathbf{k-4})_\mathbf{0}$ representation, and these are all the bosonic fields in the short multiplet.

It is easy to see that all the fields found above fit into this type of multiplet. Looking at singlets of $SU(2)_L$, we find that we have a a real scalar field in the $(\mathbf{3}, \mathbf{1})_\mathbf{0}$, a complex scalar field in the $(\mathbf{1}, \mathbf{1})_\mathbf{2}$ and a vector field in the $(\mathbf{1}, \mathbf{1})_\mathbf{0}$, which fits into the above structure with $k = 3$ (in this case the highest component of the multiplet vanishes). Similarly, the other fields fill out other representations whose lowest component is a real scalar field in the $(\mathbf{k+2}, \mathbf{k})_\mathbf{0}$ representation, and which include a complex $(\mathbf{k}, \mathbf{k})_\mathbf{2}$ scalar field, a vector $(\mathbf{k}, \mathbf{k})_\mathbf{0}$ field, and a real $(\mathbf{k-2}, \mathbf{k})_\mathbf{0}$ scalar field, for $k = 1, 2, \cdots$. The dimension of the lowest component is determined by the general formula for chiral primaries to be $\Delta = 2j + R/2 = k + 1$, and then the Q^2 components have dimension $k + 2$ and the highest component has dimension $k + 3$.

In principle, we should compute the masses of all these fields by expanding the 7-brane action, but this is not necessary since SUSY guarantees that the masses will be the ones corresponding to these dimensions. We should, however, comment on one apparent mystery, which is that we find a real scalar field in the $(\mathbf{k+2}, \mathbf{k})_\mathbf{0}$ representation with dimension $k+1$, while the $(\mathbf{k}, \mathbf{k+2})_\mathbf{0}$ field has dimension $k+5$. Naively, these fields arise from spherical harmonics of the same Laplacian eigenvalue so they should have the same mass, and therefore the operators should have the same scaling dimension. However, while the corresponding vector spherical harmonics indeed have the same eigenvalue of the Laplacian, they have eigenvalues of opposite sign of the operator $*D$ defined by $(*DY)^a = \epsilon^{abc} D_b Y_c$, where a, b, c are S^3 coordinates and D_b is the covariant derivative. This operator squares to the Laplacian, and the eigenvalues of the vector spherical harmonics in $(\mathbf{k}, \mathbf{k+2})$ representations have an opposite sign from the eigenvalues of the $(\mathbf{k+2}, \mathbf{k})$ spherical harmonics. This operator enters into the mass formula for the fields through the coupling $\int d^8x F \wedge F \wedge D^{(4)}$ in the 7-brane action, where F is the gauge field strength on the 7-brane and $D^{(4)}$ is the 4-form RR field of type IIB string theory. This leads to an equation of motion for the gauge field A of the form

$$\Box A^{A} \sim \epsilon^{ABCDEFGH} D_B A_C F^{(5)}_{DEFGH}\,, \tag{4.1}$$

where $F^{(5)}$ is the self-dual 5-form field strength of $D^{(4)}$. In the solution we are expanding around, $F^{(5)}$ is non-zero in the AdS_5-components. Thus, we find a term of the form $\Box A^a \sim \epsilon^{abc} D_b A_c$ in the equation of motion, which leads to a shift in the mass of the field corresponding to the eigenvalue of $*D$, in agreement with our results above.

Next, we would like to compare our results with the spectrum of operators in the SCFT. The only case for which this is known is the D_4 case, when the theory on the 3-branes is an $USp(2N)$ gauge theory with an anti-symmetric hypermultiplet and 4 fundamental hypermultiplets [6, 8, 9]. The scalar fields in this theory include a complex scalar field X from the vector multiplet, which is in the $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{N(2N+1)})_{\mathbf{2}}$ representation of the global and local symmetry group

$$SU(2)_R \times SU(2)_L \times SO(8) \times USp(2N) \times U(1)_R\,. \tag{4.2}$$

The hypermultiplet scalar fields are q from the fundamental hypermultiplets, in the $(\mathbf{2},\mathbf{1},\mathbf{8},\mathbf{2N})_{\mathbf{0}}$ representation, and Y from the anti-symmetric hypermultiplet, in the $(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{N(2N-1)})_{\mathbf{0}}$ representation.[2] Both of these fields obey reality conditions of the form $q^A_a = \epsilon^{AB} J_{ab} (q^\dagger)^b_B$ and $Y^{AA'}_a = \epsilon^{AB}\epsilon^{A'B'} J_{ab} (Y^\dagger)^b_{BB'}$, where A, B are $SU(2)_R$ indices, A', B' are $SU(2)_L$ indices, a, b are gauge group indices, and J is the appropriate anti-symmetric tensor. In the field theory the $SU(2)_L$ symmetry may be identified with the flavor symmetry of the anti-symmetric hypermultiplet.

We are interested here only in operators which are charged under $SO(8)$ (the other operators were described and compared with the AdS construction in [10], as described in the previous section). Obviously, these operators must involve at least two squark fields q, or the fermions ψ_q in the quark multiplets. It is easy to see that any operator with more than two quarks will be the product (or the sum of products) of more than one gauge invariant field, so it should not be compared with single-particle states in the AdS theory. Thus, it is enough to look at operators with two quark fields. The simplest such operator is just a product of two q's. Obviously, the product must be anti-symmetric in the $USp(2N)$ indices to give a gauge-invariant field, so it can either by symmetric in the $SO(8)$ indices and anti-symmetric in the $SU(2)_R$ indices or vice versa. In the first case we get a field in the $(\mathbf{1},\mathbf{1},\mathbf{35}+\mathbf{1})_{\mathbf{0}}$ of the global symmetry, but this is not a chiral primary, since it does not have any R-charge.[3] On the other hand,

[2] The $\mathbf{N(2N-1)}$ representation of $USp(2N)$ is actually reducible as $(\mathbf{N(2N-1)}-\mathbf{1})+\mathbf{1}$, with the $\mathbf{1}$ corresponding to the center of mass motion of the 3-branes along the worldvolume of the 7-branes. Since this is a decoupled free field we do not expect to see it in the bulk gravity description.

[3] Therefore, it does not obey the relation between the dimension and R-charge for a chiral operator in the free theory. Alternatively we can see that by acting on it with SUSY generators we get a field in the (larger) $(\mathbf{2},\mathbf{1},\mathbf{35}+\mathbf{1})$ representation.

in the second case we get a real scalar field in the $(\mathbf{3},\mathbf{1},\mathbf{28})_\mathbf{0}$ representation, and we can identify it with the $(\Delta = 2)$ field of the same representation that we found above. The rest of this representation can now be constructed by acting on this operator with the supercharges. The $Q\bar{Q}$ component will be exactly the global $SO(8)$ current (with $\Delta = 3$), while the Q^2 component involves terms of the form qXq and $\psi_q\psi_q$.

The other fields described above can similarly be constructed from the product of $k-1$ Y-fields with the two quarks, starting from $q_a^A Y_B^{ab} q_b^C$ where the $SU(2)_R$ indices A, B, C must be multiplied symmetrically to get a chiral field, and the $SU(2)_L \times SO(8)$ indices (in the $(\mathbf{2},\mathbf{28})$ for $k = 2$) were suppressed. It is slightly less trivial to show that these are the only chiral primary fields in the theory. Any product of X with q turns out to be a descendant because of the $W = qXq$ superpotential. Similarly, any anti-symmetric combination of Y's is a descendant because of the $W = YXY$ piece of the superpotential (the equation of motion of X enables us to replace it by two q's, allowing us to decompose the field into a product of more than one gauge-invariant field). Thus, for the D_4 case we find an exact agreement between the AdS prediction for the spectrum and the field theory results for large N, which can be viewed as evidence for the conjecture of [1]. In both cases, the spectrum of (G-charged) chiral primary fields includes the global symmetry current multiplet, and an infinite (for large N) series of copies of this multiplet, with increasing dimensions and $SU(2)_R \times SU(2)_L$ representations.

For finite N, k cannot be arbitrarily large in the field theory, since for k of order N the products of k fields are no longer independent (in particular, for $N = 1$ Y is a singlet, and only the $k = 1$ fields are independent). However, as discussed in [7], it is not clear how to see this in the AdS construction. Similar bounds were discussed for $AdS_3 \times S^3$ in [18, 19, 20]. For other groups G, the construction gives us a prediction for the spectrum of chiral primaries which it is not clear how to check directly in field theory. For $G = H_0, H_1, H_2$ we can construct the corresponding theory by flows from the D_4 theory [12], which enables us to construct the chiral primaries as subsets of the primaries of the D_4 theory. For $G = E_6, E_7, E_8$ it is not known how to construct these spectra directly from field theory; perhaps the predictions of the AdS/CFT correspondence may be tested in string theory. We find the same structure for the spectrum of G-charged states in all of these cases, with the only difference being in the global symmetry representation.

5. $\mathcal{N} = 1$ superconformal field theories from intersecting seven-branes

In this section we study the conformal field theories arising when 3-branes sit at the singularity in (2.3). We will describe the general procedure for computing the large N spectrum of these SCFTs, but we will not compute them explicitly. The results here may easily be generalized also to the case of three intersecting 7-branes.

JHEP07(1998)013

5.1. Bulk contribution

The analysis is similar to what we did above for $\mathcal{N}=2$ theories. We can choose coordinates so that the angular part of (2.3) becomes

$$ds^2 = d\theta^2 + \sin^2(\theta)d\psi^2 + \cos^2(\theta)d\phi_1^2 + \sin^2(\theta)\cos^2(\psi)d\phi_2^2 + \sin^2(\theta)\sin^2(\psi)d\phi_3^2\,, \quad (5.1)$$

with α-deformed boundary conditions and monodromies in the ϕ_1 and ϕ_2 variables. The spherical harmonics will now be labeled by the momenta m_1, m_2 and m_3 in the ϕ variables, and by two additional non-negative integers m and n, such that the eigenvalue of the Laplacian is $k(k+4)$ where $k = |m_1| + |m_2| + |m_3| + 2m + 2n$. In the S^5 case all these numbers are integers. In our case the periodicity conditions on ϕ_1, ϕ_2 are different, so $m_i = \tilde{m}_i/(1-\alpha_i/2)$ $(i=1,2)$, but m_3, n, m are still integers. The $U(1)_R$ charge in the $\mathcal{N}=1$ superconformal algebra is the sum of the $SO(2)$'s acting on ϕ_1, ϕ_2, ϕ_3, so it will be $2m_1 + 2m_2 + 2m_3$.

Of course, all these shifts affect only fields which are charged under the $SO(2)$ symmetries in the spaces transverse to the singularities. In particular, the fields which are uncharged under the R-symmetry, such as the energy-momentum tensor (the zero mode of the graviton) and the global symmetry currents (the zero modes of the 7-brane gauge fields) will not be affected, and will remain in the theory with the same dimension, as is required by their conservation equations.

5.2. Contribution from fields at singularities

In this case we have two sets of singularities (7-branes) along two three-spheres in the compact space which intersect along an S^1. So, there will be modes coming from the 7-branes and modes coming from the intersection of the two sets of 7-branes. The analysis of the 7-brane modes is similar to what we did in §4, except that now the S^3 is really replaced by a singular space since one of the angles will have a different periodicity. This can be taken into account as we did for the bulk modes in §3, but now with S^3 replacing S^5. Thus, we will again find the same spectrum described in the previous section arising from each singularity, but with shifted $U(1)_R$ charges and dimensions.

Next, we turn to the contribution of the intersection region. In many cases this region corresponds to a strongly coupled $d=6$ $\mathcal{N}=1$ fixed point theory [16], about which not much is known. However, in other cases the low-energy spectrum is known. A particularly simple case is the intersection of two D_4 singularities, which can be described as a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold of the type IIB string theory [21, 13]. The field theory on the D3-brane in this case is a $d=4, \mathcal{N}=1$ theory which, by construction, has an $SL(2,\mathbb{Z})$ electric-magnetic duality symmetry. Unfortunately, unlike the case of a single D_4 singularity described in the previous section, it is not known how to write down a Lagrangian for this theory [13, 8] so we cannot compare our results with field theory, but our results may help to find a field theory description for this theory. An intersection of this type can occur at any value of the string coupling, and in particular, at weak coupling one can perform a perturbative analysis of the spectrum and find that there is a tensor multiplet living at the intersection. This is a small multiplet of the $d = 6, \mathcal{N} = 1$ SUSY, so again it will give rise to chiral multiplets in the corresponding $d = 4, \mathcal{N} = 1$ SCFT. We will find one such multiplet for every Kaluza-Klein mode on the S^1, since the tensor multiplet lives on $S^1 \times AdS_5$. The $U(1)_R$ symmetry of the SCFT will include a contribution from the $SO(2)$ symmetry of the S^1, so the corresponding fields will have all possible integer values of the R charge. The tensor multiplet of $d = 6, \mathcal{N} = 1$ includes a real scalar, an anti-self-dual tensor field and fermions. Upon reduction on S^1 the bosonic spectrum will include the KK modes of a tensor field (or a vector field which is dual to the tensor in 5 dimensions) and a real scalar field. These fields naturally fit into a field strength multiplet W_α, whose bosonic modes include a real scalar field D and a tensor field $F_{\mu\nu}$ (the real scalar field usually appears as an auxiliary field in gauge theories, but here it is an independent primary field). We find one such multiplet for every possible value of the momentum along the S^1, up to infinity in the large N limit. The existence of these multiplets (at least for large N) is a prediction of the CFT/AdS correspondence, which may help in finding a Lagrangian formulation of the CFT in this case.

Similarly, in other intersections of singularities hypermultiplets arise at the intersection points. These are also small multiplets of the $d = 6, \mathcal{N} = (1, 0)$ SUSY algebra, and in the field theory they would correspond to the usual chiral primary multiplets, whose bosonic components are two complex scalar fields Φ and F. In other cases, the low-energy theory at the intersection of two singularities can be an interacting conformal theory, whose spectrum is not well-defined. We will not discuss these cases here.

6. Supergravity solution for 3-branes in a 7-brane background

A 3-brane probe moving in a 7-brane background has a low energy effective action which is the same as that of the exact solution of the corresponding low-energy field theories [22, 23, 28, 6]. In order to argue that the geometry is related to the field theory one needs in these cases to resort to a non-renormalization theorem which explains why the long distance result (gravity solution) can be continued to short distances (the field theory regime). Generally, this is valid only when we have at least $\mathcal{N} = 2$ SUSY. In this section we go beyond the probe approximation and we see how the 3-brane deforms the geometry. We will find that if the number of 3-branes is large the geometry can be trusted also in the field theory regime. Therefore with this solution one could calculate non-BPS processes involving a wide variety of energy regimes. We make an ansatz that reduces the problem of solving the full supergravity equations to solving the Laplace equation on the background generated by the 7-branes. We were not able to find a full solution of this Laplace equation, which in principle could be found numerically to extract information about the field theory. We will, however, discuss some qualitative features of the solution. The solution has a structure quite similar to other cases where branes can be localized within branes [24].

We now describe the solution, which boils down to a general recipe for introducing a 3-brane in a 7-brane background. We first consider a general solution of intersecting 7-branes which preserves some supersymmetry. The cases of interest are (i) A single stack of parallel 7-branes preserving 16 supercharges (or F-theory on $K3$), (ii) Two stacks of parallel 7-branes sharing 5+1 common directions and preserving a total of 8 supercharges (or F-theory on CY_3) and (iii) Three stacks of 7-branes, with any two stacks sharing 5+1 common directions and all of them sharing a total of 3+1 common directions, preserving a total of 4 supercharges (or F-theory on CY_4). Only in the first case (i) the solution is known explicitly. In the case (ii) the behaviour of the dilaton is known [13] but to our knowledge a general form of the metric is not known explicitly. In any case we will assume that we have a solution for the 7-branes by themselves. Then we will add the 3-branes. In cases (i),(ii) and (iii) the worldvolume field theory has $\mathcal{N}=2, \mathcal{N}=1$ and $\mathcal{N}=1$ supersymmetry, respectively. In case (iii) the 3-brane has to be chosen with the right orientation so that it does not break additional supersymmetries.

Let us assume that we have a supergravity solution for one of the three cases: it will be of the form

$$ds^2 = dx_{\|}^2 + g_{ij} dx^i dx^j \,, \tag{6.1}$$

where $dx_{\|}^2$ is the flat Minkowski metric in the directions 0123. Furthermore g is a Kähler metric and the complexified IIB coupling $\tau_{IIB} = \chi + i e^{-\phi}$ is a holomorphic function and describes, together with g_{ij}, an elliptically fibered CY space. Since these cases, by assumption, preserve supersymmetry, there is a spinor ϵ which satisfies

$$\begin{aligned} \delta\lambda &= \Gamma^M P_M \epsilon^* = 0 \\ \delta\psi_M &= (\partial_M + \frac{1}{4}\omega_M^{ab}\Gamma_{ab} - \frac{i}{2}Q_M)\epsilon = 0 \,, \end{aligned} \tag{6.2}$$

where we use the notation of [25]. In the cases (i), (ii) and (iii) τ depends on one, two or three complex variables, respectively. We find that the spinors satisfy the conditions $\Gamma_I \epsilon = 0$, where I runs over the complex variables z_I on which τ depends, so that we get one, two or three conditions on the spinor. All these conditions are compatible with each other, and they each break one half of the supersymmetry.

In the $\mathcal{N}=2$ case (i), the 7-brane metric is explicitly known [26]

$$g_{z\bar{z}} = \tau_2(z) \mid \eta^2(\tau(z)) \prod_{i=1}^{n} (z - z_i)^{-\frac{1}{12}} dz \mid^2 , \tag{6.3}$$

where $z = x^8 + i x^9$, z_i are the positions of the 7-branes, and $\tau(z)$ is the modular parameter of the elliptic fiber of the F-theory compactification.

Now, consider introducing 3-branes into the problem with their worldvolume spanning the $0-3$ directions. We make the following ansatz for the Einstein metric[4]

$$ds^2 = f^{-1/2} dx_{\|}^2 + f^{1/2} g_{ij} dx^i dx^j \tag{6.4}$$

and for the 5-form field

$$F_{0123i} = -\frac{1}{4}\partial_i f^{-1} \,. \tag{6.5}$$

The dilaton is the same as in the solution for the 7-branes. $f(x^i)$ is a function of the coordinates transverse to the 3-brane. The self-duality condition $F = \star_{10} F$ implies that the dual components of F_{0123i} are non-zero, and in order to be able to solve for a gauge potential we need

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g} g^{ij} \partial_j f) = -(2\pi)^4 N \frac{\delta^6(x - x^0)}{\sqrt{g}} \,, \tag{6.6}$$

where we have included a source term at the position of the N D3-branes so that we produce the right value for the flux of the five-form field strength. If the 3-branes are at different positions we just replace $N\delta(x - x^0) \to \sum_i \delta(x - x_i^0)$.

It is easy to see that the ansatz (6.4) (6.5) leads to a preserved supersymmetry. We can write the supersymmetry parameter as $\eta = f^{-1/8}\epsilon$, where ϵ is the supersymmetry preserved by the 7-branes alone. Then it is easy to see that all the supersymmetry variations, which now include also the 5-form field, vanish provided that

$$(1 + i\hat{\Gamma}_{0123})\eta = 0 \,, \tag{6.7}$$

where $\hat{\Gamma}$ refers to the flat metric gamma matrices. This constraint is automatically satisfied in case (iii) with our choice for the charge of the 3-brane (6.5), while in the other cases half of the SUSYs will be preserved. Thus, we have verified that our ansatz preserves the expected amount of supersymmetry, and this, together with (6.6), guarantees that it will also satisfy the equations of motion.

Now we should discuss more precisely the regime of validity of the supergravity solution. We will be interested here in the field theory limit, analogous to the decoupling limit described in detail in [1]. In that limit only the local F-theory geometry will be relevant, and it will encode not only the information about the low-energy effective action but (according to the conjecture [1]) the full information about the large N limit of the field theory. To describe the field theory limit we will impose the boundary condition that $f \to 0$ when $x \to \infty$, as opposed to $f \to 1$ which is relevant for the standard D3+7-branes solution. This amounts to taking the decoupling limit as in [1]. In other words, we are taking the limit $\alpha' \to 0$ keeping the mass of the 7-7, 3-7 and 3-3 strings fixed. In order to see when the gravity approximation is valid we first note that the equation (6.6) is linear, so the solution for N branes is related to the solution with $N = 1$ by $f_N = N f_1$. When we insert this into our ansatz (6.4) (6.5) we find,

[4] This ansatz appeared also in [14] while this paper was in progress.

after defining new variables $x'_{||} = x_{||}/\sqrt{N}$, that there is an overall factor of $\sqrt{N}$ in the metric so that by taking N large the curvature becomes small almost everywhere. The curvature might blow up at the position of the 7-branes, but this can be taken into account by adding the fields propagating on the 7-branes (as we did above).

We also need to understand what happens very close to the point where the 3-branes are sitting (x^0) and what happens at infinity. If x^0 is a non-singular point in the original F-theory geometry, then the solution very close to x^0 will behave as $f \sim N/(x - x^0)^4$ and the geometry locally will be $AdS_5 \times S^5$. This is just saying that we will have the $\mathcal{N} = 4$ gauge theory in the infrared. The behaviour in the ultraviolet (large x) will depend on the F-theory geometry. If the geometry is such that the dilaton asymptotes to a constant value then we also have a conformal field theory in the ultraviolet which will be like the ones we described in previous sections. This physically means that at high energies we do not distinguish whether the 7-branes and 3-branes are all together or not. Another possibility is that the dilaton goes to zero at infinity. It has already been shown in [6, 8, 9] that the behaviour of the dilaton in F-theory is the same as the behaviour of the coupling constant in the gauge theory. A novel feature of the large N limit is that the gravity description is valid in the field theory regime (though it is not *perturbative* field theory). If the dilaton goes to zero at infinity then the gravity approximation breaks down as we go to large distances from the 7-branes. This can be seen by solving (6.6) approximately for large $|z|$, and we find that $f \sim N/(|z|^4(\log|z|)^2)$. Then we see that the square of the Ricci tensor in the string frame diverges.[5] This is expected since in these cases at high energies perturbation theory becomes a good approximation and therefore gravity should fail. The distance at which the curvature diverges grows exponentially with N, mirroring the logarithmic behaviour of the coupling as a function of energy in the field theory (in a gauge theory of rank N with an N-independent one-loop beta function). At lower energies gravity will still be a good approximation if the number of 3-branes is large enough. The fact that the geometrical description fails for large distances for asymptotically free theories in 1+1 and 2+1 dimensions was discussed in [27]. In the cases discussed in [27] the running of the effective coupling was power-like and due to the engineering dimensions of the coupling. In this four-dimensional case the running of the coupling is logarithmic so it is more like what is expected for QCD. We could consider, for example, the theory that arises when we start with the $\mathbb{Z}_2$ orientifold [28] and we move the four D7-branes away. This corresponds to making the fundamental hypermultiplets infinitely massive. We get an $\mathcal{N} = 2$, $USp(2N)$ gauge theory with a hypermultiplet in the antisymmetric representation. The running of the coupling is logarithmic but independent of N, since it is given by the 7-brane solution. This agrees with the field theory expectation.

[5]It is necessary to go to the string frame to asses the validity of the gravity approximations since the dilaton is going to zero. Actually the square of the Ricci tensor goes to a constant in the Einstein frame.

We could also have a solution where we have 3-branes sitting on a D7-brane. This is a theory which at low energies has the matter content of $\mathcal{N}=4$ SYM plus a fundamental hypermultiplet. In this case the gravity solution will fail in the infrared (close to the 3-branes) since the dilaton is going to zero. This, of course, is in agreement with the fact that the field theory becomes free in the infrared in this case.

Note that the form of the metric (6.4) is such that the equations for a BPS string, or a string web [29, 31], are the same as if we neglected the metric generated by the 3-brane (and set $f=1$). Consider a configuration where the 7-branes are separated from each other, the 3-branes are separated from the 7-branes, and some 3-branes are also separated from the rest of the 3-branes. In this situation we could have strings (or string webs) going between different 3-branes, between 3-branes and 7-branes or between different 7-branes. In the gravity description the 3-7 strings become strings going between the 7-branes and the horizon $U=0$. All these types of states are then states in the field theory, and for large N the proper length of these strings is large so that we can trust the semiclassical description. This is in contrast with the situation at small N where the field theory regime and the semiclassical descriptions of the string webs do not overlap. Notice that strings going between different 7-branes can be viewed as the bound states of two quarks (hypermultiplets in the fundamental of $USp(2N)$). This can be seen starting from a quark-antiquark pair at some distance and through a description as in [30] one can see that the configuration would decay into a string going between two different 7-branes. The fact that strings going between 7-branes play a role in the field theory was demonstrated above where we found that they corresponded to operators in the 4d CFT (for example, to the global symmetry currents). At the conformal point we do not see the 3-3 strings or the 3-7 strings since they become strongly interacting massless particles. Of course the effect of these interactions is summarized by string theory on the AdS geometry.

Another interesting thing to calculate is the Kähler metric in the moduli space of the 3-branes. Consider a 3-brane separated from the rest of the 3-branes. Its dynamics are described by a Born-Infeld action in the supergravity background, which reduces at low energies to

$$\mathcal{L} \sim \int \tau_2 F^2 + \tau_1 F \wedge F + g_{ij} \partial X^i \partial X^j \,. \tag{6.8}$$

So, we see that the Kähler metric is just the metric produced by the 7-branes. This gives a method to compute the Kähler metric for $\mathcal{N}=1$ theories for N and $g_{YM}^2 N$ large. In principle one could also compute corrections in $1/\sqrt{g_{YM}^2 N}$ and $1/N$, which are α' and string loop corrections, respectively.

6.1. Three-branes in more general spaces

The ansatz that we found above, equations (6.4) and (6.5), is quite general. Indeed, we can start with *any* supergravity solution with zero B-fields (not necessarily supersymmetric) which is the product of four dimensional Minkowski space and some six-dimensional geometry (of the form (6.1)), where the dilaton field need not be a constant, but all the fields depend only on the six-dimensional coordinates. Then, the solution after we add the 3-branes has the form (6.4) (6.5) with f given by (6.6). Of course, in non-supersymmetric cases the solution that we get in this way could be corrected by α' or loop corrections. These corrections will be small in the large N limit (if we take N large with everything else kept fixed). In the decoupling limit, as above, only the local geometry will be important and $f \to 0$ when $x \to \infty$. These solutions give the large N limit of field theories that arise when 3-branes move in the corresponding geometry. In fact, we can argue that the solution must have this form since we could compactify the three spatial dimensions parallel to the 3-brane and do a U-duality transformation which takes the 3-brane charge into momentum. The solution carrying only momentum is given in terms of a single harmonic function which satisfies the Laplace equation in the transverse space (in Einstein frame) [33] (supersymmetry was not necessary for the plane wave solutions studied in [33]).

In particular, one could study 3-branes moving on an ALE space or near a singularity in a CY_3 manifold. 3-branes at various singularities and orbifold points were analyzed at the conformal point in [32, 34, 35, 36]. The above ansatz provides a way to solve for the theory away from the conformal point, after moving the 3-branes away or blowing up the A_k singularity into a smooth ALE space. In the latter case there will be strings on the AdS space coming from 3-branes wrapped on the blown-up 2-cycles, which should have some field theory interpretation (as found for other branes in [36]). Of course it is necessary again to solve the Laplace equation (6.6) on the corresponding background. All that we said in the previous section about the validity of the supergravity solution, etc., goes through also for this case. Similarly one could analyze field theories with less supersymmetry.

6.2. First order approximation

The equation (6.6) for the $\mathcal{N}=2$ case becomes

$$[g_{z\bar{z}}\nabla_y^2 + 4\partial_z\partial_{\bar{z}}]f = -(2\pi)^4 N\delta^2(z-z_0)\delta^4(y)\,, \tag{6.9}$$

where $g_{z\bar{z}}$ is given by (6.3). To illustrate an aspect of these solutions assume that z_0 is not on any 7-brane. This means that the metric is regular there and that one can find some new coordinates $\tilde{z}$ so that $\tilde{z}_0 = 0$ and such that the metric has the expansion $g = 1 + c|\tilde{z}|^2 + \cdots$. Then we can solve the equation (6.9) iteratively by writing $f = f_0 + f_1 + \cdots$, where

$$f_0 = \frac{4\pi N}{(y^2+|\tilde{z}|^2)^2}, \tag{6.10}$$

f_1 satisfies

$$[\nabla_y^2 + 4\partial_{\tilde{z}}\partial_{\bar{\tilde{z}}}]f_1 = -c|\tilde{z}|^2\nabla_y^2 f_0 \tag{6.11}$$

and so on. Solving (6.11) we find that f_1 is given by

$$f_1 = -\frac{c}{6}\frac{(y^4 + 3y^2|\tilde{z}|^2 + 6|\tilde{z}|^4)}{(y^2 + |\tilde{z}|^2)} f_0 \,, \tag{6.12}$$

so, as expected, when $\tilde{z} \to 0$ the correction becomes small. From (6.12) one might expect that the leading irrelevant correction is due to an operator of dimension six. This is not true, however, since the leading irrelevant correction will come from the fact that that the dilaton is not constant, and the first derivative of the dilaton at z_0 will give rise to an operator of dimension five in the **6** representation of $SO(6)$. This is the leading irrelevant correction to the $\mathcal{N} = 4$ low-energy theory on the 3-branes. In fact, the coefficient c above is quadratic in the parameter of this perturbation, and corresponds to other operators (starting with dimension 6) which are also induced in this background.

Of course, the equation (6.6) could also be solved numerically.

6.3. Solutions in cases of constant coupling

Consider the $\mathcal{N} = 2$ case when the coupling is constant. We will see that in this case we can solve (6.6) (generalizing our results for the conformal cases above). The metric transverse to the 7-branes will be of the form $ds^2 = g_{\bar{z}z}|dz|^2 = \tau_2|da|^2$ where a is the quantity appearing in the Seiberg-Witten solution. Thus, we can define a new variable $w(z)$ such that $dw = \sqrt{\tau_2}da$. This equation defines a (multivalued) holomorphic function. A solution of (6.6) can then be written as

$$f \sim \sum_i \frac{N}{(|w - w_0^{(i)}|^2 + y^2)^2} \,, \tag{6.13}$$

where y are the coordinates transverse to the 3-branes and parallel to the 7-branes and i runs over all images of the position of the 3-brane, z_0, i.e. all the possible values $w^{(i)} = w(z_0)$. In orbifold cases, D_4, E_n, the solutions (6.13) describe branes sitting together in the physical z plane, but of course separated from the 7-branes. In the H_n cases (6.13) leads to solutions containing several groups of branes in the physical z space. These are some solutions but not the most general solutions. Solutions that have only one group of branes in the physical z-space can be gotten by considering functions of w which, as opposed to (6.13), are not single valued functions of w. These involve the generalized spherical harmonics discussed in section 3.

Similarly we can construct solutions for other cases of constant τ which were described in [31].[6] As an example consider the case in which a D_4 singularity splits into two H_1 singularites which we set at positions $z_1 = 0$ and $z_2 = 1$. In his case $da \sim dw \sim z^{-1/4}(1-z)^{-1/4}dz$ then we see that

$$w \sim z^{3/4} F(3/4, 1/4, 7/4, z) \,, \tag{6.14}$$

[6]We thank B. Zwiebach for discussions about this.

where F is a hypergeometric function. In this case we see that for large z we get $w \sim z^{1/2}$ which is the behaviour at the D_4 singularity. It is simple to take a solution where one has threebranes at different points in the physical space by considering eqn. (6.13). One could find similar solutions in other cases with constant dilaton discussed in [31].

Acknowledgments

We would like to thank S. Kachru, D. Morrison, A. Strominger, C. Vafa and B. Zwiebach for discussions. O.A. would like to thank Harvard University and Tel-Aviv University for their hospitality during the course of this work.

As we were finishing this paper we learned that T. Hauer was pursuing ideas similar to those in section 3.

This work was supported in part by DOE grant DE-FG02-96ER40559.

References

[1] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, hep-th/9711200.

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 10 [hep-th/9802109].

[3] E. Witten, *Anti-de sitter space and holography*, hep-th/9802150.

[4] C. Vafa, *Evidence for F theory*, *Nucl. Phys.* **B 469** (1996) 403 [hep-th/9602022].

[5] D. Morrison and C. Vafa, *Compactifications of F theory on Calabi-Yau threefolds 1,2*, *Nucl. Phys.* **B 473** (1996) 74 [hep-th/9602114]; *Nucl. Phys.* **B 476** (1996) 437 [hep-th/9603161].

[6] T. Banks, M.R. Douglas and N. Seiberg, *Probing F Theory with branes*, *Phys. Lett.* **B 387** (1996) 278 [hep-th/9605199].

[7] O. Aharony, Y. Oz and Z. Yin, *M theory on $AdS_p \times S^{11-p}$ and superconformal field theories*, hep-th/9803051, to appear in *Phys. Lett.* **B**.

[8] O. Aharony, J. Sonnenschein, S. Yankielowicz and S. Theisen, *Field theory questions for string theory answers*, *Nucl. Phys.* **B 493** (1997) 177 [hep-th/9611222].

[9] M.R. Douglas, D.A. Lowe and J.H. Schwarz, *Probing F theory with multiple branes*, *Phys. Lett.* **B 394** (1997) 297[hep-th/9612062].

[10] A. Fayyazuddin and M. Spalinski, *Large N superconformal gauge theories and supergravity orientifolds*, hep-th/9805096.

[11] P.C. Argyres and M.R. Douglas, *New phenomena in $SU(3)$ supersymmetric gauge theory*, *Nucl. Phys.* **B 448** (1995) 93 [hep-th/9505062].

[12] P.C. Argyres, M.R. Plesser, N. Seiberg and E. Witten, *New $\mathcal{N} = 2$ superconformal field theories in four dimensions*, *Nucl. Phys.* **B 461** (1996) 71 [hep-th/9511154].

[13] A. Sen, *A nonperturbative description of the Gimon-Polchinski orientifold*, *Nucl. Phys.* **B 489** (1997) 139 [hep-th/9611186].

[14] A. Kehagias, *New type IIB vacua and their F-theory interpretation*, hep-th/9805131.

[15] K. Dasgupta and S. Mukhi, *F-theory at constant coupling*, *Phys. Lett.* **B 385** (1996) 125 [hep-th/9606044].

[16] M. Bershadsky and A. Johansen, *Colliding singularities in F theory and phase transitions*, *Nucl. Phys.* **B 489** (1997) 122 [hep-th/9610111].

[17] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, *Mass spectrum of chiral ten dimensional $N = 2$ supergravity on S^5*, *Phys. Rev.* **D 32** (1985) 389.

[18] J. Maldacena and A. Strominger, *AdS_3 black holes and a stringy exclusion principle*, hep-th/9804085.

[19] E. Martinec, *Matrix models of AdS gravity*, hep-th/9804111.

[20] S. Gubser, *Can the effective string see higher partial waves?*, *Phys. Rev.* **D 56** (1997) 4984 [hep-th/9704195].

[21] J.D. Blum and A. Zaffaroni, *An orientifold from F theory*, *Phys. Lett.* **B 387** (1996) 71 [hep-th/9607019].

[22] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087].

[23] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $\mathcal{N} = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099].

[24] J. Gauntlett, *Intersecting branes*, APCTP lectures, hep-th/9705011.

[25] M. Green, J. Schwarz and E. Witten, *Superstring theory*, Cambridge University Press (1987).

[26] B. Greene, A. Shapere, C. Vafa and S.T. Yau, *Stringy cosmic strings and noncompact Calabi-Yau manifolds*, *Nucl. Phys.* **B 337** (1990) 1.

[27] N. Itzhaki, J. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large N limit of theories with sixteen supercharges*, *Phys. Rev.* **D 58** (1998) 046004 [hep-th/9802042].

[28] A. Sen, *F-theory and orientifolds*, *Nucl. Phys.* **B 475** (1996) 562 [hep-th/9605150].

[29] A. Sen, *BPS states on a three brane probe*, *Phys. Rev.* **D 55** (1997) 2501 [hep-th/9608005];
A. Johansen, *A comment on BPS states in F-theory in 8 dimensions*, *Phys. Lett.* **B 395** (1997) 36 [hep-th/9608186];
A. Fayyazuddin, *Results in supersymmetric field theory from 3-brane probe in F-theory*, *Nucl. Phys.* **B 497** (1997) 101 [hep-th/9701185];
M. Gaberdiel and B. Zwiebach, *Exceptional groups from open strings*, *Nucl. Phys.* **B 518** (1998) 151 [hep-th/9709013];
I. Kishimoto and N. Sasakura, *M-theory description of BPS string in 7-brane background*, hep-th/9712180;
O. Bergman and A. Fayyazuddin, *String junctions and BPS states in Seiberg-Witten theory*, hep-th/9802033;
Y. Imamura, *E_8 flavor multiplets*, hep-th/9802189;
A. Mikhailov, N. Nekrasov and S. Sethi, *Geometric realizations of BPS states in $\mathcal{N}=2$ theories*, hep-th/9803142;
T. Hauer, *Equivalent string networks and uniqueness of BPS states*, hep-th/9805076;
O. De Wolfe, T. Hauer, A. Iqbal and B. Zwiebach, *Constraints on the BPS spectrum of $\mathcal{N}=2$ $d=4$ theories with ADE flavor symmetry*, hep-th/9805220;
O. Bergman and A. Fayyazuddin, *String junction transitions in the moduli space of $\mathcal{N}=2$ SYM*, hep-th/9806011.

[30] J. Maldacena, *Wilson loops in large N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002];
S-J. Rey and J. Yee, *Macroscopic strings as heavy quarks in large N gauge theory and anti-de-Sitter supergravity*, hep-th/9803001.

[31] M.R. Gaberdiel, T. Hauer and B. Zwiebach, *Open string - string junction transitions*, hep-th/9801205.

[32] S. Kachru and E. Silverstein, *4d conformal field theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855 [hep-th/9802183].

[33] G. Horowitz and A. Tseytlin, *On exact solutions and singularities in string theory*, *Phys. Rev.* **D 50** (1994) 5204 [hep-th/9406067] and references therein.

[34] M. Bershadsky and Z. Kakushadze, *String expansion as large N expansion of gauge theories*, *Nucl. Phys.* **B 523** (1998) 59 [hep-th/9803076];
A. Lawrence, N. Nekrasov and C. Vafa, *On conformal field theories in four-dimensions*, hep-th/9803015;
Z. Kakushadze, *Gauge theories from orientifolds and large N limit*, hep-th/9804184; M. Bershadsky and A. Johansen, *Large N limit of orbifold field theories*, hep-th/9803249.

[35] Y. Oz and J. Terning, *Orbifolds of $AdS_5 \times S^5$ and 4d conformal field theories*, hep-th/9803167.

[36] E. Witten, *Baryons and branes in anti-de Sitter space*, hep-th/9805112.